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Effects of an external magnetic field on thermo-acoustical waves in a linear isotropic thermo-elastic dielectric material

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SUMMARY

Thermo-mechanico-electromagnetic coupled waves propagating in a linear isotropic thermo-elastic dielectric material are theoretically investigated, in case an external magnetic field is applied to the material. Here the constitutive equations derived from the Clausius–Duhem inequality and Vernotte's heat conduction law are adopted. There are three types of coupled waves: the predominantly electromagnetic wave, the predominantly mechanical transverse wave and the predominantly thermo-mechanical longitudinal wave. The first and second waves have no thermal coupling. The third wave has thermal coupling and its propagation velocity and attenuation constant are perturbed by the external magnetic field.

1. Introduction

For a polarizable, non-conducting, non-magnetic, deformable elastic dielectric, Toupin [1] proposed a remarkable dynamic phenomenological theory, and derived a unified mathematical theory of the piezoelectric, photoelastic properties, the Faraday effect, and magneto-elastic dragging of the elastic dielectric material. Basing on Toupin's theory, McCarthy [2] and McCarthy and Green [3] analysed the propagation and growth of plane acceleration waves in a hyperelastic dielectric material in an external magnetic field. Tokuoka and Kobayashi [4] investigated the mechanico-electromagnetic coupled wave in a linear isotropic elastic dielectric material, where the suppressed magnetic field was assumed to have any direction relative to the propagation direction.

On the other hand, thermal effects on elastic waves have been studied widely. Among the numerous literature, Thurston [5] discussed in detail plane harmonic waves propagating in an elastic conductor of heat, where the Fourier's law:

$$\boldsymbol{q} = -\kappa \operatorname{grad} T \tag{1.1}$$

was employed, where $\boldsymbol{q}$ is the heat flux, κ is the conductivity. As a consequence of it, the temperature field is governed by a parabolic equation and then a thermal disturbance propagates with an infinite velocity. In order to remedy this unpleasant feature, Vernotte [6] proposed a modified Fourier's law:

$$\dot{\boldsymbol{q}} = -\frac{1}{\tau}(\boldsymbol{q} + \kappa \operatorname{grad} T), \tag{1.2}$$

where the superposed dot denotes the material time derivative and τ is a relaxation time. Making use of the modified law Tokuoka [7, 8] investigated the propagation and growth of plane acceleration waves in an isotropic linear thermo-elastic material. It was shown that there exist two purely mechanical shear waves with a velocity v_S and two thermo-longitudinal waves with velocities v_{TL}:

$$v_{TL}^2 = v_L^2[1 + \tfrac{1}{2}[(\beta^2 + \gamma - 1) \pm \{(\beta^2 + \gamma - 1)^2 + 4\gamma\}^{\frac{1}{2}}]], \tag{1.3}$$

and a damping constant ν_{TL} given by

$$\nu_{TL} = \frac{\beta^2}{2} \frac{\{(v_{TL}/v_L)^2 - 1\}^2}{(v_{TL}/v_L)^2[\gamma + \{(v_{TL}/v_L)^2 - 1\}^2]}, \tag{1.4}$$

where

$$v_S = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}, \quad v_L = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}} \tag{1.5}$$

denote, respectively, the purely mechanical transverse and longitudinal wave velocities and

$$\beta^2 = \frac{\kappa}{(\lambda + 2\mu)c_V\tau}, \quad \gamma = \frac{(3\lambda + 2\mu)^2\alpha^2 T_0}{\rho(\lambda + 2\mu)c_V} \tag{1.6a, b}$$

are dimensionless material constants, where λ and μ are the Lamé elastic constants, α is the coefficient of thermal expansion, c_V is the specific heat at constant volume and ρ is the density.

This paper has two purposes, one is to investigate the propagation of plane infinitesimal thermo-mechanico-electromagnetic coupled waves in a linear isotropic thermo-elastic dielectric material, and the other is to reveal the effects of an external magnetic field on thermo-acoustical waves.

2. Basic equations

The electromagnetic field in polarizable, non-conducting, non-magnetic, deformable media is governed by the following system of equations:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{E} = 0, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{2.1a}$$

$$\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = 0, \qquad \operatorname{div} \boldsymbol{D} = 0 \tag{2.1b}$$

with

$$\boldsymbol{H} = \mu_0^{-1}\boldsymbol{B} + \dot{\boldsymbol{x}} \times \boldsymbol{P}, \qquad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{2.2}$$

The vectors $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{P}$ are the electric field, magnetic flux density, electric flux density, magnetic field and polarization, respectively. $\boldsymbol{x}$ is the position vector of the particle. The fundamental constants μ_0 and ε_0 are related to the speed of light in vacuum by $\mu_0\varepsilon_0 = c^{-2}$. In addition to (2.1), let us assume an equation of molecular equilibrium:

$$\mathscr{E} + {}_L\boldsymbol{E} = 0, \tag{2.3}$$

which was introduced by Toupin [9], where ${}_{\mathrm{L}}\boldsymbol{E}$ is the local field and

$$\mathscr{E} = \boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B} \tag{2.4}$$

is the effective field intensity.

The conservation laws of mass and linear momentum are expressed as

$$\frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} = 0, \tag{2.5}$$

$$\rho \ddot{x}^i = \sigma^{ij}_{,j} + f^i, \tag{2.6}$$

where σ^{ij}, f^i are, respectively, symmetric Cauchy stress and body force per unit mass. Here and henceforth, a rectangular Cartesian coordinate system is employed, and a comma followed by a suffix denotes the partial derivative with respect to a coordinate. Now we consider the body force originating in the electric polarization, that is,

$$\boldsymbol{f} = -(\operatorname{div} \boldsymbol{P})\mathscr{E} + \overset{*}{\boldsymbol{P}} \times \boldsymbol{B}, \tag{2.7}$$

where

$$\overset{*}{\boldsymbol{P}} = \frac{\partial \boldsymbol{P}}{\partial t} + \dot{\boldsymbol{x}} \operatorname{div} \boldsymbol{P} + \operatorname{curl}(\boldsymbol{P} \times \dot{\boldsymbol{x}}) \tag{2.8}$$

is the convected time derivative of $\boldsymbol{P}$. The balance law of energy for a dielectric material in the absence of free charge, current, magnetization and heat supply is given by

$$\rho(\psi + T\eta)^{\cdot} = \sigma^i_j \dot{x}^j_{,i} + \overset{*}{P}{}^i \mathscr{E}_i - q^i_{,i}, \tag{2.9}$$

where ψ and η are the specific Helmholtz free energy and entropy, respectively [10]. In order to complete the field equations, the heat conduction law (1.2) must be assumed.

Now consider waves propagating in a thermo-elastic dielectric in an external constant magnetic field $\boldsymbol{B}_0$. Then we can put

$$\boldsymbol{E} = \boldsymbol{e}, \quad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}, \quad \boldsymbol{P} = \boldsymbol{p}, \quad \boldsymbol{x} = \boldsymbol{u} \tag{2.10a, b, c, d}$$

where $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{p}$ and $\boldsymbol{u}$ denote the wave fields and the suffix zero denotes a quantity evaluated in equilibrium. The wave fields are assumed to be weak and the dimensionless temperature $\theta = (T - T_0)/T_0$ to be small and then the second and higher order terms of them and their derivatives may be neglected.

Let us take the constitutive assumptions:

$$\psi = \hat{\psi}(\varepsilon_{km}, \theta, p_k), \quad \sigma_{ij} = \hat{\sigma}_{ij}(\varepsilon_{km}, \theta, p_k), \tag{2.11a, b}$$

$$\eta = \hat{\eta}(\varepsilon_{km}, \theta, p_k), \quad {}_{\mathrm{L}}e_i = {}_{\mathrm{L}}\hat{e}_i(\varepsilon_{km}, \theta, p_k), \tag{2.11c, d}$$

where $\varepsilon_{km} = (u_{k,m} + u_{m,k})/2$ is the infinitesimal strain tensor, and let us consider the Clausius-Duhem inequality:

$$\rho\dot{\eta} \geqq -\frac{1}{T} q^i_{,i} + \frac{1}{T^2} q^i T_{,i} \tag{2.12}$$

which must be satisfied for all admissible processes [11] and is reduced to

$$\rho(\dot{\psi} + \eta\dot{T}) - \sigma^{i}_{j}\dot{x}^{j}_{,i} + {}_{\mathrm{L}}e^{i}\dot{p}_{i} + \frac{1}{T}q^{i}T_{,i} \leqq 0. \tag{2.13}$$

From (2.11a) and (2.13) we can conclude that

$$\sigma^{ij} = \rho\frac{\partial\hat{\psi}}{\partial\varepsilon_{ij}}, \quad \eta = -\frac{1}{T_0}\frac{\partial\hat{\psi}}{\partial\theta}, \quad {}_{\mathrm{L}}e_i = -\rho\frac{\partial\hat{\psi}}{\partial p^i}, \tag{2.14a, b, c}$$

$$q^{i}T_{,i} \leqq 0. \tag{2.15}$$

The free energy may be approximated by the quadratic form:

$$\rho\psi = \tfrac{1}{2}C^{ijkm}\varepsilon_{ij}\varepsilon_{km} + C^{ij}\varepsilon_{ij}\theta + \tfrac{1}{2}C\theta^2 + S^{ij}_{k}\varepsilon_{ij}p^k + S_i\theta p^i + \tfrac{1}{2}\chi^{-1}_{ij}p^ip^j, \tag{2.16}$$

where C^{ijkm}, C^{ij}, C, S^{ij}_{k}, S_i and χ^{-1}_{ij} are material constants and have the following symmetry relations:

$$C^{ijkm} = C^{jikm} = C^{ijmk} = C^{kmij}, \quad C^{ij} = C^{ji}, \tag{2.17a, b}$$

$$S^{ij}_{k} = S^{ji}_{k}, \quad \chi_{ij} = \chi_{ji}. \tag{2.17c, d}$$

Substituting (2.16) into (2.14), we have

$$\sigma^{ij} = C^{ijkm}\varepsilon_{km} + C^{ij}\theta + S^{ij}_{k}p^k, \tag{2.18}$$

$$\eta = -\frac{1}{\rho T_0}(C^{ij}\varepsilon_{ij} + C\theta + S_ip^i), \tag{2.19}$$

$$-{}_{\mathrm{L}}e_i = S^{km}_{i}\varepsilon_{km} + \chi^{-1}_{ij}p^j + S_i\theta. \tag{2.20}$$

Since we assume an isotropic material, we have

$$C^{ijkm} = \lambda\delta^{ij}\delta^{km} + \mu(\delta^{ik}\delta^{jm} + \delta^{im}\delta^{jk}), \tag{2.21a}$$

$$C^{ij} = -\rho(3\lambda + 2\mu)\alpha T_0\delta^{ij} \equiv -\Delta\delta^{ij}, \tag{2.21b}$$

$$C = -\rho c_{\mathrm{V}}T_0, \tag{2.21c}$$

$$S^{ij}_{k} = S_i = 0, \tag{2.21d}$$

$$\chi^{-1}_{ij} = (\varepsilon_0\chi)^{-1}\,\delta_{ij}, \tag{2.21e}$$

where χ is the polarizability constant and Δ is the thermo-elastic coupling constant.

3. Plane harmonic waves and propagation condition

Let us consider a plane harmonic wave with attenuation constant a, wave number k, frequency ω and propagation direction $\boldsymbol{n}$, that is,

$$F = \bar{F}\exp(\Gamma\boldsymbol{n}\cdot\boldsymbol{x} - i\omega t), \tag{3.1}$$

$$\Gamma = a + ik \tag{3.2}$$

where F stands for the small deviation fields u_i, e_i, b_i, p_i, q_i and θ, $\bar{F}$ denotes the corresponding complex amplitude and a is related to the damping constant ν defined by Tokuoka [7] through $a = -\nu/(v\tau)$. Now assume

$$a \ll k, \tag{3.3}$$

which means that the attenuation of wave amplitude is negligibly small within a propagation distance compared to a wave-length. Refer to Thurston [5].

From (1.2), (2.1), (2.3), (2.6) and (3.1) we have

$$-\rho\omega^2\bar{u}_i = \{\mu\bar{u}_i + (\lambda + \mu)n_i n_j \bar{u}^j\}\Gamma^2 - i\omega e_{ijk}\bar{p}^j B_0^k - \rho\Delta\Gamma n_i\bar{\theta}, \tag{3.4}$$

$$\bar{e} - i\omega(\bar{u} \times B_0) - (\varepsilon_0\chi)^{-1}\bar{p} = 0, \tag{3.5}$$

$$\frac{\Gamma}{\mu_0}(n \times \bar{b}) + i\omega\varepsilon_0\bar{e} + i\omega\bar{p} = 0, \tag{3.6}$$

$$\Gamma(n \times \bar{e}) - i\omega\bar{b} = 0, \tag{3.7}$$

$$n\cdot(\varepsilon_0\bar{e} + \bar{p}) = 0, \quad n\cdot\bar{b} = 0, \tag{3.8a, b}$$

$$-i\omega\bar{q} = -\frac{1}{\tau}(\bar{q} + \kappa T_0\Gamma n\bar{\theta}), \tag{3.9}$$

where e_{ijk} is a permutation tensor. From (2.14), (2.19) and (2.21) the energy equation (2.9) gives

$$\Gamma n\cdot\bar{q} = i\omega\rho\Delta\Gamma n\cdot\bar{u} + i\omega\rho c_V T_0\bar{\theta}. \tag{3.10}$$

Let us assume that the wave propagates alongt he x_3-axis and the magnetic field has an arbitrary direction specified by direction cosines (l_1, l_2, l_3). Then we have

$$n = (0, 0, 1), \quad B_0 = (B_0 l_1, B_0 l_2, B_0 l_3). \tag{3.11a, b}$$

By (3.6), (3.7) and (3.11a), we obtain

$$\bar{e} = \left(\frac{V^2}{\varepsilon_0(N^2 - V^2)}\bar{p}_1, \frac{V^2}{\varepsilon_0(N^2 - V^2)}\bar{p}_2, -\frac{1}{\varepsilon_0}\bar{p}_3\right), \tag{3.12}$$

$$\bar{b} = \left(-\frac{NV^2}{\varepsilon_0 v(N^2 - V^2)}\bar{p}_2, \frac{NV^2}{\varepsilon_0 v(N^2 - V^2)}\bar{p}_1, 0\right), \tag{3.13}$$

where $v = \omega/k$ is the wave velocity, $V = v/c$ is the dimensionless velocity and $N \equiv \equiv (-i\Gamma/k) = 1 - i(a/k)$. The electromagnetic field deviations (3.12) and (3.13) satisfy the relation (3.8) identically. From (3.4), (3.9) and (3.11) we have in case of $v \neq v_S$ and $v \neq v_L$

$$\bar{u} = \left(\frac{iB_0}{\rho\omega}\frac{V^2}{V^2 - V_S^2N^2}(\bar{p}_2 l_3 - \bar{p}_3 l_2), \frac{iB_0}{\rho\omega}\frac{V^2}{V^2 - V_S^2N^2}(\bar{p}_3 l_1 - \bar{p}_1 l_3),\right.$$
$$\left.\frac{iB_0}{\rho\omega}\frac{V^2}{V^2 - V_L^2N^2}(\bar{p}_1 l_2 - \bar{p}_2 l_1) + \frac{i\Delta}{\omega v}\frac{V^2}{V^2 - V_L^2N^2}N\bar{\theta}\right), \tag{3.14}$$

where $V_S \equiv v_S/c$ and $V_L \equiv v_L/c$, and

$$\bar{q} = \left(0, 0, \frac{(\kappa T_0/\tau)N}{v[1 + (i/\tau\omega)]}\bar{\theta}\right). \tag{3.15}$$

In the last section the uncoupled waves with velocities v_S and v_L will be discussed.

Eliminating $\bar{e}$, $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{q}}$ in (3.5) and (3.10) by (3.12), (3.14) and (3.15), we can obtain a set of linear homogeneous equations with respect to the polarization amplitude $\bar{\boldsymbol{p}}$ and the dimensionless temperature amplitude $\bar{\theta}$,

$$\boldsymbol{Q}(V^2, \boldsymbol{l})\mathfrak{p} = 0 \tag{3.16}$$

where $\mathfrak{p}$ is a four-dimensional vector with components $(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{\theta})$ and the symmetric 4×4 matrix $\boldsymbol{Q}(V^2, \boldsymbol{l})$ is given after some lengthy manipulations as follows:

$$\begin{pmatrix} \dfrac{(1+\chi)V^2-N^2}{N^2-V^2}+\dfrac{Al_3^2V^2}{V_S^2N^2-V^2}+\dfrac{Al_2^2V^2}{V_L^2N^2-V^2} & -\dfrac{Al_1l_2V^2}{V_L^2N^2-V^2} & -\dfrac{Al_3l_1V^2}{V_S^2N^2-V^2} & \rho\Delta\cdot\dfrac{Al_2}{B_0v}\dfrac{V^2}{V_L^2N^2-V^2}N \\ \cdot & \dfrac{(1+\chi)V^2-N^2}{N^2-V^2}+\dfrac{Al_3^2V^2}{V_S^2N^2-V^2}+\dfrac{Al_1^2V^2}{V_L^2N^2-V^2} & -\dfrac{Al_2l_3V^2}{V_S^2N^2-V^2} & -\rho\Delta\cdot\dfrac{Al_1}{B_0v}\dfrac{V^2}{V_L^2N^2-V^2}N \\ \cdot & \cdot & -\left\{(1+\chi)-\dfrac{A(l_1^2+l_2^2)V^2}{V_S^2N^2-V^2}\right\} & 0 \\ \cdot & \cdot & \cdot & \dfrac{\rho A}{B_0^2}\left\{\rho c_V T_0-\dfrac{\kappa T_0}{\tau v^2}\left(1+\dfrac{i}{\tau\omega}\right)^{-1}N^2+\dfrac{\rho\Delta^2}{v^2}\dfrac{V^2}{V_L^2N^2-V^2}N^2\right\} \end{pmatrix}, \tag{3.17}$$

where

$$A = \frac{\varepsilon_0\chi B_0^2}{\rho} \tag{3.18}$$

is a dimensionless quantity depending on the material constants and the suppressed magnetic intensity.

For a wave to exist, the amplitude $\mathfrak{p}$ must not be a null vector and then from (3.16) we have the propagation condition:

$$\det\boldsymbol{Q}(V^2, \boldsymbol{l}) = 0, \tag{3.19}$$

which gives, in general, six propagation velocities for a given direction of the external magnetic field. Substituting a solution V^2 of (3.19) into (3.16) we have an amplitude $\mathfrak{p}$. The ratio of the amplitude can be given by the ratio of the cofactor with respect to any row elements of $\boldsymbol{Q}$, that is,

$$\bar{p}_1 : \bar{p}_2 : \bar{p}_3 : \bar{\theta} = \hat{Q}_{\alpha 1} : \hat{Q}_{\alpha 2} : \hat{Q}_{\alpha 3} : \hat{Q}_{\alpha 4} \tag{3.20}$$

for any $\alpha = 1, 2, 3, 4$, where $\hat{Q}_{\alpha\beta}$ indicates a cofactor of an element $Q_{\alpha\beta}$. By substitution of (3.20) into (3.12), (3.13) and (3.14), we can obtain the ratios of the amplitudes of $\bar{\boldsymbol{e}}$, $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{u}}$.

In practice A is a very small quantity, *e.g.*, it is approximately 10^{-9} for $B_0 = 10^4$ Gauss, $\chi = 10$ and $\rho = 8 \times 10^3$ Kg/m^3. Thus we may suppose that, while waves in a non-vanishing magnetic field must be, in general, thermo-mechanico-electromagnetic coupled waves, they consist of the predominantly electromagnetic waves, the predominantly mechanical transverse waves and the predominantly thermo-mechanical longitudinal waves, and their propagation velocities deviate from the values of $c/\sqrt{1+\chi}$, v_S and v_L about $O(A)$ in the first approximation.

To begin with, let us consider the predominantly electromagnetic waves. The small quantities V_S^2 and V_L^2 can be neglected in comparison with V^2, the (1.4) and (2.4) elements of $\boldsymbol{Q}$ become $O(1/c)$, and the (4.4) element becomes $\rho A c_V T_0/B_0^2 + O(1/c^2)$. Thus the thermoelastic coupling constant Δ has no effect on the propagation condition. In other words, the predominantly electromagnetic waves suffer no thermal influence in our approximation, and so the propagation condition is reduced to the one treated by Tokuoka and Kobayashi [4] and the attenuation constant a vanishes.

In the case of predominantly mechanical transverse waves, V^2 can be considered to be a small quantity and we can estimate that $V_S^2 - V^2 \sim O(A)$. Then the (1.4), (2.4) and (4.4) elements of $\boldsymbol{Q}$ become of higher order than the other elements. The thermo-elastic coupling constant Δ, therefore, does not appear in the propagation condition, and the predominantly mechanical transverse waves do not suffer any thermal effect within the assumed approximation.

For the discussion of the predominantly electromagnetic waves and the predominantly mechanical transverse waves, we refer to Tokuoka and Kobayashi [4].

4. Predominantly thermo-mechanical longitudinal waves

In the case of no electromagnetic effect, the propagation condition (3.19) gives

$$\rho c_V T_0 - \frac{\kappa T_0}{\tau v^2}\left(1 + \frac{i}{\tau\omega}\right)^{-1} N^2 + \frac{\rho \Delta^2}{v^2}\,\frac{V^2}{V_L^2 N^2 - V^2}\,N^2 = 0. \tag{4.1}$$

Then, if there is an electromagnetic effect, the left-hand side of (4.1) can be estimated to be $O(A)$. So we have the propagation condition as follows:

$$\begin{vmatrix} O(1)+O(A) & O(A) & O(A) & O(A) \\ \cdot & O(1)+O(A) & O(A) & O(A) \\ \cdot & \cdot & O(1)+O(A) & O(A) \\ \cdot & \cdot & \cdot & O(A^2) \end{vmatrix} = 0. \tag{4.2}$$

If we retain the second-order terms of A and neglect the third- and higher-order terms in the determinant (4.2) and also neglect V^4 in comparison with V^2, we obtain the complex propagation condition:

$$\left\{\rho c_V T_0\left(1+\frac{i}{\tau\omega}\right)-\frac{\kappa T_0}{\tau v^2}N^2\right\}(v_L^2N^2-v^2)^2$$
$$+\rho\Delta^2N^2\left(1+\frac{i}{\tau\omega}\right)\{v_L^2N^2-v^2+A(l_1^2+l_2^2)v^2\}=0, \tag{4.3}$$

where the relations $v=cV$ and $v_L=cV_L$ have been used.

The real part of (4.3) gives

$$\left(\rho c_V T_0-\frac{\kappa T_0}{\tau v^2}\right)(v_L^2-v^2)^2+\rho\Delta^2\{v_L^2-v^2+A(l_1^2+l_2^2)v^2\}=0 \tag{4.4}$$

in virtue of the basic assumption (3.3). Then we have

$$\left(\frac{v}{v_L}\right)^4-(1+\beta^2+\gamma)\left(\frac{v}{v_L}\right)^2+\beta^2-\gamma\left(\frac{v}{v_L}\right)^2A(l_1^2+l_2^2)\frac{(v/v_L)^2}{1-(v/v_L)^2}=0, \tag{4.5}$$

where β^2 and γ were defined in (1.6). Equation (4.5) may be reduced to the equation derived by Tokuoka and Kobayashi [4] when A is equal to zero. The solutions of (4.5) are

$$v=v_{TL}\left[1\pm\frac{(v_{TL}/v_L)^2}{1-(v_{TL}/v_L)^2}\frac{A(l_1^2+l_2^2)\gamma}{\{(\beta^2+\gamma-1)^2+4\gamma\}^{\frac{1}{2}}}\right]^{\frac{1}{2}} \tag{4.6}$$

within the first order of A, where v_{TL} denotes the velocity of the pure thermo-longitudinal wave and is given by (1.3), and the double signs in (4.6) and (1.3) must be taken in the same order. The second term in the bracket in the right hand side of (4.6) represents the effects of the external magnetic field.

Now consider the imaginary part of (4.3). Using the assumption (3.3), we obtain

$$2a\left[2v_L^2\left(\rho c_V T_0-\frac{\kappa T_0}{\tau v^2}\right)+\rho\Delta^2\left\{1+\frac{A(l_1^2+l_2^2)v^2}{v_L^2-v^2}+\frac{v_L^2}{v_L^2-v^2}\right\}-\frac{\kappa T_0}{\tau v^2}(v_L^2-v^2)\right]$$
$$=\frac{\rho c_V T_0}{\tau v}(v_L^2-v^2)+\frac{\rho\Delta^2}{\tau v}\left\{1+\frac{A(l_1^2+l_2^2)}{v_L^2-v^2}\right\}, \tag{4.7}$$

which gives the attenuation constant:

$$a=-\frac{1}{\tau v}\frac{\beta^2}{2}\frac{\{(v/v_L)^2-1\}^2}{(v/v_L)^2[\gamma+\{(v/v_L)^2-1\}^2]}$$
$$\times\left[1+\frac{2A(l_1^2+l_2^2)\gamma}{\gamma+\{(v/v_L)^2-1\}^2}\frac{(v/v_L)^2}{(v/v_L)^2-1}\right], \tag{4.8}$$

within the first order of A. When the external magnetic field vanishes, we have the attenuation constant for thermo-acoustical waves obtained by Tokuoka [7]:

$$a_{TL}=-\frac{\nu_{TL}}{\tau v_{TL}}. \tag{4.9}$$

The effects of the external magnetic field appear in (4.8) not only through the perturbed term but also through the velocity relation (4.6).

From (4.6) and (4.8) we can say that *the influence of the external magnetic field on the propagation velocities and the attenuation constant of thermo-acoustical waves is determined by* $(l_1^2 + l_2^2)$ *which is the square of the component of the projection of the vector* $\boldsymbol{l}$ *on the plane normal to the propagation direction.*

When the suppressed magnetic field is parallel to the propagation direction, i.e., $l_1 = l_2 = 0$ and $l_3 = 1$, we have

$$v = v_{\mathrm{TL}}, \qquad a = a_{\mathrm{TL}}. \tag{4.10a, b}$$

This means that the thermo-acoustical waves may propagate without being influenced by the electromagnetic field if the direction of the external magnetic field coincides with the wave propagation direction.

The ratios of amplitudes can be obtained easily. From (3.20) we have to the first order of A

$$\bar{p}_1 : \bar{p}_2 : \bar{p}_3 : \bar{\theta} = 0 : 0 : 0 : 1. \tag{4.11}$$

In order words, the polarization amplitude $\bar{\boldsymbol{p}}$ is of higher order than the dimensionless temperature amplitude $\bar{\theta}$. Similarly we have

$$\frac{\bar{\boldsymbol{p}}}{\bar{\theta}}, \frac{\bar{\boldsymbol{e}}}{\bar{\theta}}, \frac{\bar{\boldsymbol{b}}}{\bar{\theta}} = O(A). \tag{4.12}$$

Substituting (4.11) into (3.14), we have

$$\frac{\bar{u}_1}{\bar{\theta}}, \frac{\bar{u}_2}{\bar{\theta}} = O(A) \tag{4.13}$$

and by assumption (3.3)

$$\frac{(v/v_{\mathrm{L}})^2 - 1}{(v/v_{\mathrm{L}})^2} = \frac{\Delta}{v_{\mathrm{L}}} \frac{\bar{\theta}}{(-i\omega\bar{u}_3)}, \tag{4.14}$$

which is identical with the amplitude ratio given by Tokuoka [7] for thermo-acoustical waves except that v in (4.14) is not v_{TL} but is given by (4.6).

Then we can say that within the order A (i) *electromagnetic fields over the constant external magnetic field are not induced by the propagation of the predominantly thermo-mechanical longitudinal waves* and (ii) *the suppressed magnetic field influences the ratio of the amplitude only through the change of the propagation velocity.*

5. Uncoupled waves

In Sec. 3 we assumed that $v \neq v_{\mathrm{S}}$ and $v \neq v_{\mathrm{L}}$. Here we discuss the possibility of existence of waves having the propagation velocities v_{S} and v_{L}.

Let us consider the case of $v = v_{\mathrm{S}}$ and $a = 0$. From (3.4) we have

$$\bar{p}_2 l_3 - \bar{p}_3 l_2 = 0, \qquad \bar{p}_3 l_1 - \bar{p}_1 l_3 = 0,$$
$$\frac{v_{\mathrm{L}}^2 - v_{\mathrm{S}}^2}{v_{\mathrm{S}}^2} \bar{u}_3 + \frac{iB_0}{\rho\omega} (\bar{p}_1 l_2 - \bar{p}_2 l_1) + \frac{i\Delta}{\omega v_{\mathrm{S}}} \bar{\theta} = 0. \tag{5.1a, b, c}$$

Eliminating $\bar{\boldsymbol{q}}$ from (3.9) and (3.10), we obtain

$$\left\{\frac{\kappa T_0}{\tau} - \rho v_S^2\left(1 + \frac{i}{\tau\omega}\right)c_V T_0\right\}\bar{\theta} = i\omega\left(1 + \frac{i}{\tau\omega}\right)\rho\Delta v_S \bar{u}_3. \tag{5.2}$$

Substituting (3.12) into (3.5) we have

$$\begin{aligned} &-\bar{p}_1 - \frac{i\rho\omega}{B_0}Al_3\bar{u}_2 + \frac{i\rho\omega}{B_0}Al_2\bar{u}_3 = 0,\\ &-\bar{p}_2 + \frac{i\rho\omega}{B_0}Al_3\bar{u}_1 - \frac{i\rho\omega}{B_0}Al_1\bar{u}_3 = 0,\\ &-(1+\chi)\bar{p}_3 - \frac{i\rho\omega}{B_0}Al_2\bar{u}_1 + \frac{i\rho\omega}{B_0}Al_1\bar{u}_2 = 0. \end{aligned} \tag{5.3a, b, c}$$

Then we can easily conclude that a wave with amplitude

$$\bar{\boldsymbol{u}} \neq \boldsymbol{0}, \quad \bar{u}_3 = 0, \quad \bar{\boldsymbol{p}} = \boldsymbol{0}, \quad \bar{\theta} = 0 \tag{5.4}$$

may exist, if and only if

$$\bar{u}_1 : \bar{u}_2 = l_1 : l_2, \quad l_3 = 0. \tag{5.5}$$

In the case of $v = v_L$ and $a = 0$, we can conclude by a similar argument that a wave with amplitude

$$\bar{u}_3 \neq 0, \quad \bar{u}_1 = \bar{u}_2 = 0, \quad \bar{\boldsymbol{p}} = \boldsymbol{0}, \quad \bar{\theta} = 0 \tag{5.6}$$

may exist, if and only if

$$l_1 = l_2 = 0, \quad \Delta = 0. \tag{5.7}$$

Thus we can say that (i) *if the suppressed magnetic field is perpendicular to the propagation direction, purely mechanical transverse waves exist, where the wave oscillates along the magnetic field* and (ii) *the purely mechanical longitudinal wave can not exist unless the thermoelastic coupling constant vanishes even if the suppressed magnetic field is parallel to the propagation direction.*

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